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A note on forward wakes in rotating fluids

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The two studies by Professor Miles (1970*a*, *b*) on the motion of a rotating fluid past a body raise the important question of the determinancy of such flows, by theoretical arguments, which it seems worth while making more precise. Suppose we have a fluid which when undisturbed has a uniform velocity U in the direction Ox and a uniform angular velocity Ω about Ox. It is slightly disturbed, the resulting motion having velocity components $(u+U, v, \Omega r+w)$ relative to cylindrical polar axes (x, r, θ), centre O and in which r measures distance from Ox, while θ is the azimuthal angle. Assuming that u, v, w are sufficiently small for their squares and products to be neglected, and are independent of θ, the equations governing their behaviour reduce to

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x} &= -\frac{\partial P}{\partial x}+\nu\nabla^2 u,\\ \frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}-2\Omega w &= -\frac{\partial P}{\partial r}+\nu\left[\nabla^2 v-\frac{v}{r^2}\right],\\ \frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+2\Omega v &= \nu\left[\nabla^2 w-\frac{w}{r^2}\right],\\ u=\frac{1}{r}\frac{\partial\psi}{\partial r},\quad v &= -\frac{1}{r}\frac{\partial\psi}{\partial x},\end{aligned}\right\}\tag{1}$$

where ψ is the stream function, P is the reduced pressure and ν the kinematic viscosity. Interest mainly centres on the form of the solution when $\nu=0$ and $t\to\infty$, i.e. the steady motion of an inviscid fluid. On setting $\partial/\partial t=0$, $\nu=0$ the equations (1) reduce to

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\alpha^2\psi = F(r),\quad w=\frac{\alpha}{r}\psi+G(r),\tag{2}$$

where $\alpha=2\Omega/U$ and F, G are arbitrary functions of r. If we now think of the disturbance as being caused by a surface $r=f(x)$, $x_1<x<x_2$, bounding a convex region, where $f(x_1)=f(x_2)=0$ and f takes its maximum value, equal to δ, at $x=0$, then the linearization holds in two cases: (*a*) when $\alpha\delta\gg 1$, f arbitrary, and (*b*) when $|f'|\ll 1$, $\alpha\delta$ arbitrary. In neither case, however, can the solution be completed. For example, if $|f'|\ll 1$ the boundary condition on the body reduces to

$$\psi=-\tfrac{1}{2}Ur^2\quad\text{when}\quad r=f(x),\tag{3}$$

and we do not know the appropriate condition on ψ at large distances from the body. Even if we knew that $F = G = 0$ and $\psi \to 0$ as $|x| \to \infty$, (3) is insufficient to determine ψ as we need an analogue of the radiation condition to complete the solution.

Two methods have been suggested to overcome these difficulties. First, we can suppose that $\nu = 0$ and the disturbed motion begins from a state of relative rest at $t = 0$. Then the Laplace transform $\bar{\psi}(x, r, s)$ of the stream function ψ, with respect to time using s as parameter, can be written in the form

$$r\int_0^\infty \mathscr{A}(\beta; s, \alpha)\, e^{lx} J_1(\beta r)\, d\beta, \tag{4}$$

if $x < 0$, where $\mathscr{A}$ is arbitrary and $l(>0)$ a known function of β, s, α. If $x > 0$ the corresponding expression for $\bar{\psi}$ consists of the sum of three terms like (4), but with $l < 0$. The properties of these general solutions have been examined by Stewartson (1958) and Trustrum (1964). On letting $s \to 0$ we can hopefully obtain the limit structure of ψ as $t \to \infty$ and deduce the properties of F, G. This was done explicitly by Stewartson (1968), who found, as a consequence, that at points in the fluid

$$\left.\begin{aligned} \psi &= \Psi + r\int_0^\alpha A(\beta)\, J_1(r\beta)\, d\beta, \\ w &= -(\alpha/r)\,\Psi + r\int_0^\alpha \beta A(\beta)\, J_1(r\beta)\, d\beta, \end{aligned}\right\} \quad \text{if} \quad x < 0; \tag{5}$$

$$\left.\begin{aligned} \psi &= \Psi - r\int_\alpha^\infty A(\beta)\, J_1(r\beta)\, d\beta + r\int_0^\infty B(\beta)\, J_1(r\beta)\, d\beta, \\ w &= -(\alpha/r)\,\Psi - r\int_\alpha^\infty \beta A(\beta)\, J_1(r\beta)\, d\beta - r\int_0^\infty \beta B(\beta)\, J_1(r\beta)\, d\beta, \end{aligned}\right\} \quad x > 0. \tag{6}$$

Here
$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \alpha^2 \Psi = 0 \tag{7}$$

and Ψ also satisfies Long's hypothesis (Long 1953), namely that it contains no wave-like terms when x is large and negative. Further $A(\beta)$, $B(\beta)$ are arbitrary functions of β except that ψ, w as defined by (5), (6) must be continuous at $x = 0$ for all $r > \delta$. The last condition may be achieved if

$$A - B = \beta\int_0^\delta a(r)\, J_1(r\beta)\, dr, \quad A + B = \int_0^\delta b(r)\, J_1(r\beta)\, d\beta, \tag{8}$$

where a and b are arbitrary functions of r in $0 < r < \delta$.

Secondly, we can suppose that $\nu > 0$, the motion is steady and undisturbed at infinity. Then the general form for ψ is a combination of integrals like (4) with s replaced by ν and different functions $\mathscr{A}$, l. Some properties of these functions have also been determined by Trustrum (1964) when ν is small, and in a private communication Professor Miles has pointed out that, for fixed x, the general solution for ψ is identical with (5), (6) in the limit $\nu \to 0$. There is no contradiction with the assumption of undisturbed motion at infinity for the terms independent of x in (5), (6) die away when $|\nu x| \gg 1$.

The result of these two approaches may be applied formally to both problems (*a*) and (*b*). In (*a*) the unsteady treatment given by Stewartson (1952) leads to the same solution provided we set $\Psi \equiv 0$. [We note that if $\Psi \not\equiv 0$ the solution must contain violent oscillations since $\alpha\delta \gg 1$.] In particular for a sphere

$$A = B = -\frac{2U}{\pi}\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right). \tag{9}$$

In case (*b*) when $\alpha = 0$, $\psi = \Psi$ except in the lee of the body [$x > 0, r < \delta$]. Further when $\alpha(x_2 - x_1) \sim 1$ the contributions to ψ, w from a and b are relatively small except in the lee. Of more interest is the flow when $\alpha\delta \sim 1$ when the contributions from a and b to ψ, w may then be significant. This may most easily be seen by choosing non-dimensional co-ordinates based on δ, when α is replaced by $\alpha\delta$ and the length of the surface becomes $(x_2 - x_1)/\delta$. Even though A and B may be as important as Ψ the boundary conditions satisfied by Ψ are independent of them. For since the perturbation velocities are small (3) may be replaced by

$$\partial\Psi/\partial x = -Uf(x)f'(x) \quad \text{when} \quad r = f(x), \tag{10}$$

so that Ψ, which satisfies Long's hypothesis, is fully determinate and independent of the wake-like terms.

An attempt to estimate A and B has been made by Stewartson (1968), who assumed that (1) holds even for fat bodies with $\alpha\delta \sim 1$ and ensured uniqueness by imposing the no-slip condition. In view of the essential contradiction between this assumption and that of small disturbances from which (1) is deduced, it is not surprising that the attempt was only partially successful from a physical standpoint. Certain features of the observations are reproduced however which suggest that viscosity does play a role in providing the missing boundary conditions. In the preceding paper Miles argued that in unseparated flow $A = B = 0$ but this is not the case when $\alpha\delta \gg 1$, where strong forward and rear wakes are set up by inviscid action alone and the notion of separation plays no part. Further, so long as the disturbances are small and $\alpha\delta \sim 1$ the only effect of A and B is a slight modification to the axial velocity and none to the direction of the streamlines. The most noticeable modifications produced by A and B are to the pressure distribution on the body and to the far field which is now no longer uniform.

It has also been suggested that for fat bodies Long's hypothesis holds for a range of values of $\alpha\delta$, including $\alpha\delta = 0$. The consequences for a sphere have been worked out (Stewartson 1958, 1969) and lead to drag coefficients which become excessively high at moderate values of $\alpha\delta$ (e.g. $C_D = 14{\cdot}39$ at $\alpha\delta = 3$). Miles (1969) then made the hypothesis that this assumption breaks down at the first onset of closed streamlines in the flow, which for a sphere occurs when $\alpha\delta \sim 2{\cdot}2$ at the place $x = 0$, $\alpha r = 2\pi$, and suggested that more accurate calculations would move it farther downstream. The occurrence of such a rotor would seem to be only remotely connected with a wake at large distances upstream of the body. In a further private communication Professor Miles points out that at $\alpha\delta \sim 2{\cdot}2$ there is also an incipient rotor at the forward stagnation point, which we shall discuss further below. Very recently Benjamin (1970) has made a study of the impulse function of a swirling unsteady inviscid flow inside a tube of radius R.

He demonstrates that if $U < 0{\cdot}522R\Omega$ it is impossible for the flow to remain undisturbed for *all* time at an arbitrarily large but fixed distance upstream if the flow near the body ultimately becomes steady. The implication is that in an unbounded fluid, which is under consideration here, there is always a forward wake independently of whether separation has taken place; little information about its structure may be deduced however.

Thus it seems that the purely theoretical approach has reached an impasse; no convincing way has yet been found of making the solution unique and at the same time relevant to experimentally realizable flows. The functions $a(r)$ and $b(r)$ seem to be non-zero but few of their properties are known. This non-uniqueness has been noted in certain problems of magnetohydrodynamics and in stratified flows: indeed it is formally present in all linearized studies of flow past finite bodies. For example a and b are indeterminate even when $\alpha = 0$ but, as mentioned earlier, only modify the flow in the lee of the body.

Actual experiments are few. Taylor (1922) was the first to observe a strong upstream wake ahead of a sphere when $\alpha\delta > \sim 2\pi$, and Long (1953) was partly led to his hypothesis, mentioned earlier, by observing the wave structure round moving bodies, and found that the column occurs only if $\alpha\delta > \sim 8$. I first became aware of the possibility of a forward wake at all $\alpha\delta > 0$ from seeing some unpublished experiments by Professor H. B. Squire and very recently Professor T. Maxworthy has made a quantitative study of this phenomenon, when the body is a sphere of radius δ. He finds that a bubble of almost stagnant fluid [i.e. with low axial velocity] always occurs ahead of the sphere and its length L increases as ν decreases. As $\nu \rightarrow 0$, L approaches a finite limit if $\alpha\delta < 4$. At the smallest value of $\alpha\delta$ used ($\sim 0{\cdot}6$), $L \approx 0{\cdot}2\delta$: at $\alpha\delta \approx 2$, $L \approx \frac{3}{2}a$ and thereafter it rapidly increases with $\alpha\delta$. Thus Miles' assumption of the validity of Long's hypothesis for $\alpha\delta < 2{\cdot}2$ is at variance with these experiments since it implies that there are no rotors at the forward stagnation point. There is also firm evidence of a forward wake, beyond the bubble. For example at $\alpha\delta \approx 1{\cdot}75$ the flow is strongly disturbed at $x = -8\delta$ but it is not clear what happens as $\nu \rightarrow 0$. Downstream the wake is present at all values of $\alpha\delta$ of course but its character changes as $\alpha\delta$ increases from zero. The ejected boundary-layer fluid tends to concentrate on the axis and there is evidence of vortex breakdown. The drag coefficient, for small values of ν, is non-zero when $\alpha\delta = 0$, falls a little as $\alpha\delta$ increases and then rises steadily, ultimately being $\approx \frac{5}{2}\alpha\delta$. It is quite different from that obtained by Stewartson (1958, 1969) on the assumption of Long's hypothesis and about 50 % more than predicted by Stewartson (1952) when $\alpha\delta \gg 1$.

Pritchard (1969) has carried out a related set of experiments in which a body moves along the axis of a rotating fluid. He concludes that an ever-lengthening column of fluid is trapped in front of the body if $\alpha\delta > \sim 3$, that at lower values of $\alpha\delta$ a finite disturbance extends far upstream and as $\alpha\delta \rightarrow 0$ a potential-flow pattern is approached. The differences between the two experiments are of degree rather than character, and may be due to the rather higher Reynolds numbers in Pritchard's experiments, but both support the view that a and b are non-zero.

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